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N-62-4-3

# 278 261

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MEMORANDUM
RM-3192-PR
JUNE 1962

# COMPOUND SIMPLE GAMES I:
# SOLUTIONS OF SUMS AND PRODUCTS

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# PREFACE

The mathematical research presented in this Memorandum is concerned with solutions for certain types of games of strategy. The theory of games is of great importance because of its applicability to a variety of conflict situations — economic, political, and military.

## SUMMARY

This is an investigation of the solutions of the games that are formed by combining two or more simple games, played by separate groups of individuals. A simple game is one that is completely specified by its winning coalitions. Two forms of combination are studied in detail in this Memorandum: the sum, in which a coalition wins if it wins in either component, and the product, in which a coalition wins only if it wins in both components. In both cases, relationships are established between the solutions of the compound games and those of the components.

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# COMPOUND SIMPLE GAMES I:
## SOLUTIONS OF SUMS AND PRODUCTS

## 1. INTRODUCTION AND DEFINITIONS

An n—person game is called "simple" provided every coalition of players either can win outright or is completely defeated. A simple game can therefore be described by listing its winning coalitions. In this paper we consider the "compound" simple games that are obtained by combining two or more simple games (with different players) into a single, larger game. Two methods of combination will be treated: the product, in which a winning coalition must contain winning contingents from each component game, and the sum, in which a winning coalition need have winning strength in just one component game. Our object, in both cases, will be to establish relationships between the stable—set solutions of the compound games and those of their components.

A verbal statement of our first two theorems will help convey the nature of the results we obtain. We find that in a compound game (sum or product) there is a solution that distributes the proceeds within each component according to a pattern that would be stable (i.e., a solution) in that component alone. The amounts must be scaled down, however, to allow for what the players in the other component(s) receive. In the case of a sum, the allocation among the different components is arbitrary, but constant. In the case of a product, all possible intergroup allocations occur in the same solution.

These theorems, then, give us an easy way to construct solutions to compound games when we know solutions to the component games. It may be noted that the sums, which have many more winning coalitions than the corresponding products, have smaller (i.e., more determinate) solutions.

The later sections of this paper contain extensions and converses of these basic results, together with the solutions of some illustrative games. In a subsequent paper we shall take up a more general class of composition rules for simple games in which the components play the role of players in a supergame, and obtain somewhat similar results.

An elementary introduction to the theory and applications of simple games will be found in [4].

Simple games. We shall denote a simple game by the symbol $\Gamma(P, \mathcal{W})$, where $P$ is a finite set (the players) and $\mathcal{W}$ is a collection of subsets of $P$ (the winning coalitions). We shall require of $\mathcal{W}$ that it include $P$ and exclude $O$ (the empty set), and that it be monotonic in the following sense:

(1) $$S \in \mathcal{W}, \ T \supseteq S \rightarrow T \in \mathcal{W}.$$

In view of (1), the collection of minimal winning coalitions, denoted by $\mathcal{W}^m$, is sufficient to specify the game. If $\mathcal{W}$ has the further property that

(2) $$S \in \mathcal{W} \rightarrow P-S \notin \mathcal{W},$$

which implies (with (1)) that every two winning coalitions intersect, then the game is said to be <u>proper</u>. If (2) fails for some S, then the game is termed <u>improper</u>. The characteristic function of an improper game in the classical theory is not superadditive.

If $\mathcal{W}$ enjoys the converse property:

(3) $$S \notin \mathcal{W} \rightarrow P-S \in \mathcal{W},$$

which implies that there are no "blocking" coalitions, then the game is said to be <u>strong</u>. The proper strong games are "constant—sum" in the classical theory; they comprise the class of simple games that was originally defined and investigated by von Neumann and Morgenstern (see [5], Chapter X).

If we count the winning coalitions, we find that in a strong game

$$|\mathcal{W}| \geq 2^{|P|-1},$$

while in a proper game

$$|\mathcal{W}| \leq 2^{|P|-1}.$$

The fewer the winning coalitions in a game, the "weaker" it is said to be.

Sums and Products. A pair of simple games with different players can be combined into a larger game in two natural ways, related to the Boolean operations of addition and multiplication.

Let $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = O$. Then the sum

$$\Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$$

is defined as the game $\Gamma(P, \mathcal{W})$ where $\mathcal{W}$ consists of all $S \subseteq P$ such that either $S \cap P_1 \in \mathcal{W}_1$ or $S \cap P_2 \in \mathcal{W}_2$. In terms of minimal winning coalitions we have

$$\mathcal{W}^m = \mathcal{W}_1^m \cup \mathcal{W}_2^m;$$

hence $|\mathcal{W}^m| = |\mathcal{W}_1^m| + |\mathcal{W}_2^m|$.

Similarly, the product

$$\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$$

is defined as the game $\Gamma(P, \mathcal{W})$, where $\mathcal{W}$ consists of all $S \subseteq P$ such that both $S \cap P_1 \in \mathcal{W}_1$ and $S \cap P_2 \in \mathcal{W}_2$. In terms of minimal winning coalitions we have

$$\mathcal{W}^m = \{S \mid S \cap P_1 \in \mathcal{W}_1^m, \ S \cap P_2 \in \mathcal{W}_2^m\};$$

hence $|\mathcal{W}^m| = |\mathcal{W}_1^m| \cdot |\mathcal{W}_2^m|$.

We see that products are always weaker than the corresponding sums. In general sums are improper and products are nonstrong.[1] Addition and multiplication are commutative and associative, and yield unique decompositions of compound games. A further discussion of these operations will be found in [4].

We may remark that games in which some player $v$ has a veto, i.e.,

$$S \in \mathcal{W} \rightarrow v \in S, \ \text{but} \ \{v\} \notin \mathcal{W},$$

have as a factor the one-person game $\Gamma(\{v\}, \ \{\{v\}\})$, which we denote by $B_1^v$. Except for the trivial "dictator" games, these are the only simple games having outcomes that cannot be blocked by any coalition.[2]

Imputations. Let $A_P$ denote the simplex of real nonnegative vectors $x$ such that $\Sigma_P \ x_i = 1$. These vectors are traditionally called "imputations"

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[1] This fact may explain why these operations do not appear in von Neumann and Morgenstern's account [5], which is limited to the constant—sum (i.e., strong and proper) case. (See below, p. 14.)

[2] Such games, in the general theory, are called weak, and the set of unblockable outcomes is called the core. A core outcome in a simple game is always the victory of a dictator or veto player, or a victory shared among several veto players.

when **P** is the set of players in a simple game. By convention, we shall say that $x_j = 0$ for any and all $j \notin P$. This is tantamount to having $A_P$ embedded in a simplex $A_Q$, where $Q$ is some large, un—specified set including all the players that might ever come under con—sideration.

Let us write $x(S)$ for $\Sigma_S x_i$. Let $R_S x$ be the "restriction" of $x$ to $S$, thus:

$$(R_S x)_i = \begin{cases} x_i & \text{if } i \in S, \\ \\ 0 & \text{if } i \notin S. \end{cases}$$

The <u>barycentric projection</u> of $x$ on $A_S$ is then given by

$$B_S x = \frac{1}{x(S)} R_S x.$$

This is not well defined if $x(S) = 0$; otherwise it is an imputation in $A_S$.

Let $S$ and $T$ be fixed, disjoint sets, and let $X$ and $Y$ be subsets of $A_S$ and $A_T$, respectively. Given $0 \leq \alpha \leq 1$, we define $X \underset{\alpha}{\times} Y$ to be the set of all imputations $z$ such that

$$z = \alpha x + (1 - \alpha)y \text{ for some } x \in X, \ y \in Y.$$

Thus we have $X \underset{1}{\times} Y = X$ and $X \underset{0}{\times} Y = Y$. For intermediate $\alpha$ values $X \underset{\alpha}{\times} Y$ is a scaled—down cartesian product of $X$ and $Y$. (In Fig. 1, the shaded rectangle represents the set $X \underset{1/3}{\times} Y$.)

Fig. I—— Illustrating the operations $\underset{\alpha}{\times}$ and $\times$

Finally, we define an operation $\times$ by

$$X \times Y = \bigcup_{0 \le \alpha \le 1} X \underset{\alpha}{\times} Y.$$

This set consists of all imputations $z$ such that $B_S z$ and $B_T z$ are in X and Y respectively, together with the sets X and Y themselves. It is sometimes called the "join" of X and Y, being the union of all line segments running between X and Y. (If X and Y are convex, as in the figure, then $X \times Y$ is the convex hull of $X \cup Y$.)

## 2.   COMPOSITION OF SOLUTIONS:   TWO BASIC THEOREMS

In this section we shall state and prove two theorems on the composition of solutions — one dealing with sums, the other with products.

We recall that a <u>solution</u> of the game $\Gamma(P, \mathcal{W})$ is a set $X$ of imputations such that

$$X = A_P - \text{dom } X.$$

Here "dom $X$" denotes the set of all $y$ such that, for some $x \in X$, the set $\{i \,|\, x_i > y_i\}$ is an element of $\mathcal{W}$. The notations "$\text{dom}_1 X$," "$\text{dom}_2 X$" will be used for domination with respect to special classes $\mathcal{W}_1, \mathcal{W}_2$.

Solutions are known to exist for all simple games. For example, $A_S$ is a solution whenever $S = \mathcal{W}^m$.

<u>The basic theorems.</u>    In the following it is always assumed that the sets $P_1$, $P_2$ have no members in common.

THEOREM 1.  If $X_i$ is a solution of $\Gamma(P_i, \mathcal{W}_i)$, $i = 1, 2$, then $X_1 \times X_2$ is a solution of $\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$.

THEOREM 2.  If $X_i$ is a solution of $\Gamma(P_i, \mathcal{W}_i)$, $i = 1, 2$, and if $0 \leq \alpha \leq 1$, then $X_1 \underset{\alpha}{\times} X_2$ is a solution of $\Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$.

The reader may now wish to read again the paragraph in Sec. 1 that gives a verbal statement of the content of these theorems. As indicated by the wording there, both theorems generalize directly to sums or products of more than two components.

If we set $\alpha = 0$ or $1$ in Theorem 2, we see that any solution of a component also solves the sum. The existence of such solutions, which completely discriminate against the rest of the players, follows from the fact that any component of a sum is a "winning fraction" of the whole game, in the sense of [3].

Note also that the solutions obtained by choosing different values of $\alpha$ in Theorem 2 are all proper subsets of the solution given in Theorem 1, based on the same subsolutions $X_1$, $X_2$. This is in accord with the intuitive idea that a weaker game should have a less determinate outcome.

<u>Proof of Theorem 1</u>. Let $\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$ and let $X = X_1 \times X_2$. (A) External stability. The first part of the proof is devoted to showing that $X \cup \operatorname{dom} X = A_P$. Let $y$ be an arbitrary point in $A_P$, and let $\beta_1 = y(P_1)$ and $\beta_2 = y(P_2) = 1 - \beta_1$. Let $y_i$ denote[3] the barycentric projection of $y$ on $A_{P_i}$, $i = 1, 2$, if this is well defined, i.e., if $\beta_i > 0$. There are a number of cases; our object each time will be to show that either $y \in X$ or $y \in \operatorname{dom} X$.

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[3]Henceforth we shall avoid direct reference to the components of vectors, and use subscripted symbols $x_i$, $y_i$, etc., to denote vectors also. Usually (but not always) they will be related in some stated way to their unsubscripted counterparts.

<u>Case 1</u>: $0 < \beta_1 < 1$.

<u>Case 1a</u>: $y_1 \in X_1$, $y_2 \in X_2$. Here $y \in X$.

<u>Case 1b</u>: $y_1 \notin X_1$, $y_2 \notin X_2$. Find a $z_1 \in X_1$ and a $z_2 \in X_2$ such that $y_1 \in \mathrm{dom}_1 z_1$ and $y_2 \in \mathrm{dom}_2 z_2$. Define the imputation $z$ by

$$z = \beta_1 z_1 + \beta_2 z_2.$$

Then it is clear that $y \in \mathrm{dom}\, z$. Moreover, $z \in X$. Hence $y \in \mathrm{dom}\, X$.

<u>Case 1c</u>: $y_1 \in X_1$, $y_2 \notin X_2$. Find a $z_2 \in X_2$ such that $y_2 \in \mathrm{dom}_2 z_2$. Denote by $S_2$ the set of players on which $z_2 - y_2$ is positive; we have $S_2 \in \mathcal{W}_2$. Choose $\epsilon > 0$ so small that $(\beta_2 - \epsilon)z_2 - \beta_2 y_2$ is also positive on $S_2$. Let $u_1$ be any interior point of $A_{P_1}$, that is, a vector that is positive on $P_1$ and zero on $P_2$, and has $u(P) = 1$. Then we assert that the vector $x$, defined by

$$x = \beta_1 y_1 + \epsilon u_1 + (\beta_2 - \epsilon)z_2,$$

is an imputation, since it is a weighted average of three imputations, and that it dominates $y$, since $x - y$ is positive on the winning set $P_1 \cup S_2$. There are now two possibilities. If it happens that $x \in X$ then we are finished with this case, since we have $y \in \mathrm{dom}\, X$. Suppose then that $x \notin X$. Then it must be the projection on $A_{P_1}$ that fails to be in $X_1$, since the other projection is in $X_2$ by construction. Hence we

can find a $z_1 \in X_1$ such that $B_{P_1} x \in \text{dom}_1 z_1$. Let $S_1 \in \mathcal{W}_1$ be the set of players on which $z_1 - B_{P_1} x$ is positive. Then the imputation $z$, defined by

$$z = (\beta_1 + \varepsilon)z_1 + (\beta_2 - \varepsilon)z_2,$$

dominates $y$, since $z - y$ is positive on the winning set $S_1 \cup S_2$. Moreover, $z \in X$. Hence again $y \in \text{dom } X$.

Case 1d: $y_1 \notin X_1$, $y_2 \in X_2$. Like Case 1c.

Case 2: $\beta_1 = 0$.

Case 2a: $y_2 \in X_2$. Then $y \in X$.

Case 2b: $y_2 \notin X_2$. The argument of Case 1c may be repeated here, with the understanding (since $y_1$ is no longer well defined) that $\beta_1 y_1$ is to be taken equal to zero wherever it appears.

Case 3: $\beta_1 = 1$. Like Case 2. This completes the proof of external stability.

(B) Internal stability. It remains to show that $X \cap \text{dom } X = O$. Suppose the contrary. Then imputations $x, y \in X$ exist with $x - y$ positive on some winning coalition $S$. Without loss of generality we may assume that $y(P_1) \geq x(P_1)$. But $x(P_1) \geq x(S \cap P_1) > y(S \cap P_1) \geq 0$. Hence the barycentric projections of both $x$ and $y$ on $A_{P_1}$ are well defined; denote them by $x_1$ and $y_1$, respectively. As a result of our assumed inequality, the difference $x_1 - y_1$ is positive on at least the set

$S \cap P_1$, which is winning in $\Gamma(P_1, \mathcal{W}_1)$. Hence $y_1 \in \mathrm{dom}_1 x_1$. But $x_1$ and $y_1$ are both elements of the "subsolution" $X_1$; we have therefore reached a contradiction. This completes the proof of Theorem 1.

<u>Proof of Theorem 2</u>. Let $\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$ and let $X^\alpha = X_1 \underset{\alpha}{\times} X_2$. (A) External stability. Let $y$ be an arbitrary point in $A_P$, with $\beta_i$ and $y_i$, $i = 1, 2$, defined as in the previous proof. Without loss of generality we may assume that $\beta_1 \le \alpha$. Let $u_1$ be an arbitrary interior point of $A_{P_1}$, and construct an imputation $z$:

$$z = \beta_1 y_1 + (\alpha - \beta_1)u_1 + (1 - \alpha)x_2,$$

where $x_2$ is an arbitrary element of the subsolution $X_2$, and the term $\beta_1 y_1$ is understood to be zero if $y_1$ is not well defined. There are three cases to be distinguished at this juncture.

<u>Case 1</u>: $z \notin X^\alpha$. Then $\alpha > 0$, and the barycentric projection of $z$ on $A_{P_1}$ is well—defined; call it $z_1$. Then we have

$$z_1 = \frac{\beta_1}{\alpha}\, y_1 + \frac{\alpha - \beta_1}{\alpha}\, u_1.$$

If $z_1 \in X_i$, then we would have $z \in X^\alpha$, contrary to assumption; hence $z_1 \notin X_1$. Hence $z_1 \in \mathrm{dom}_1 X_1$, and there exists $x_1 \in X_1$ such that $x_1 - z_1$ is positive over some $S_1 \in \mathcal{W}_1$. Therefore, multiplying by $\alpha$, we see that $\alpha x_1 - \beta_1 y_1$ is also positive over $S_1$. But $S_1$ is also winning in the full game; hence the imputation $x$, defined by

$$x = \alpha x_1 + (1 - \alpha)x_2,$$

dominates $y$. Since $x \in X^\alpha$, we have $y \in \text{dom } X^\alpha$.

__Case 2__: $z \in X^\alpha$, $\beta_1 < \alpha$. Then $z - y$ is positive on the winning coalition $P_1 \in \mathcal{W}$, and we have $y \in \text{dom } X^\alpha$.

__Case 3__: $z \in X^\alpha$, $\beta_1 = \alpha$. Then $y_1 \in X_1$ (or is undefined). If $y_2 \in X_2$ (or is undefined) as well, then $y$ is an element of $X^\alpha$, and we are through. On the other hand, if $y_2 \notin X_2$ we can reverse the roles of $P_1$ and $P_2$ from the beginning, and the new $z$ that will be constructed will not belong to $X^\alpha$ — i.e., Case 1 will apply. This completes the proof of (A).

(B) Internal stability. Suppose $x, y \in X^\alpha$, with $x - y$ positive on some winning coalition $S \in \mathcal{W}$. Then either $S \cap P_1 \in \mathcal{W}_1$ or $S \cap P_2 \in \mathcal{W}_2$; without loss of generality, assume the former. Then $\alpha = x(P_1) > 0$, so that the barycentric projections $x_1$ and $y_1$ of $x$ and $y$ on $A_{P_1}$ are well defined. Obviously $x_1 - y_1$ is positive on $S \cap P_1$; hence $y_1 \in \text{dom}_1 x_1$. This contradicts the internal stability of the subsolution $X_1$, and completes the proof of Theorem 2.

## 3.  THE SOLUTIONS OF SUMS:  A COMPLETE CHARACTERIZATION

The first two theorems gave sufficient conditions for a set of imputations to be a solution of a compound simple game. It is natural to inquire whether the conditions are necessary as well — whether every solution is of one of the types described. The answer, curiously enough, is "yes" for sums, "no" for products. We shall consider first the case of sums.

THEOREM 3. Let  X  be any solution of the sum

$$\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2).$$

Then there exist solutions  $X_1$  and  $X_2$  of  $\Gamma(P_1, \mathcal{W}_1)$  and  $\Gamma(P_2, \mathcal{W}_2)$,

respectively, and a real number  $0 \leq \alpha \leq 1$, such that  $X = X_1 \underset{\alpha}{\times} X_2$.

$\bullet$

Connection with the decomposition theory of von Neumann and

Morgenstern.  Taken together, Theorems 2 and 3  completely charac—

terize the solutions of sums.  There is a striking resemblance here to the

main result in [ 5 ] , Chapter IX, concerning solutions of "decomposable"

games.  They, too, are the cartesian products of the solutions of the

components, although a slight modification of the solution concept (the

introduction of "excess") is required at the component level to make things

fit.  Corresponding to our parameter  $\alpha$,  there is a "transfer" parameter

in the von Neumann—Morgenstern result which can be chosen arbitrarily

between certain limits, the limits depending on the possible values for the

excess in the component games.

Because of this similarity it is worth emphasizing that the two theories

are actually quite separate in application.  The composition of two essential

simple games (in the sense of [ 5 ]) is never a simple game, since some

coalitions are only half winning.  On the other hand, the sum of two simple

games (in our sense) is not even a game in the classical theory, since it

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has disjoint winning coalitions.[4]

A theoretical link between the two operations, however, is provided by an equivalence relation due to Gillies,[5] whereby the solutions of any nonsuperadditive (i.e., improper) game can be shown to be the same as the solutions of its minimum superadditive majorant, with a certain negative excess imposed. As it happens, the minimum superadditive majorants of our sums turn out to be the von Neumann–Morgenstern compositions of the same components.[6] Thus, an alternative proof of our Theorems 2 and 3 might be built around the known results in [1] and [5]. We have chosen to give direct proofs for several reasons, cne being the fact that the theory of decomposable games in [5] was presented only for the constant-sum case.

Proof of Theorem 3. We shall first show that $X$ is a cartesian product — i.e., a set of the form $X_1 \underset{\alpha}{\times} X_2$ for some $X_1 \subseteq A_{P_1}$, $X_2 \subseteq A_{P_2}$, and $0 \leq \alpha \leq 1$. Then we shall show that $X_1$ and $X_2$ are solutions of their respective subgames. Take any $x, y \in X$, and let

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[4]Note that although sums are improper, they are nevertheless useful in the analysis of proper games. For example, a game of the form $(G \oplus H) \otimes J$ may be proper. (See below, p. 20.)

[5]See [1], 66 ff.

[6]The excess that must be applied to $G$, the composition of $G_1, \ldots, G_m$, to make its solutions match those of $G_1 \oplus \ldots \oplus G_m$, may be taken to be any number $e$ that satisfies $-|G|_1 < e \leq -|G|_1 + \min_i |G_i|_1$ (notation from [5], pages 368–9). This condition ensures that every nonflat coalition of $G$ is effective everywhere in the imputation space.

$x_i$, $y_i$ denote their barycentric projections on the face $A_{P_i}$, $i = 1, 2$, if well defined. Let $u_1$, $u_2$ be arbitrary interior points of $A_{P_1}$, $A_{P_2}$, respectively. Define $\alpha_i = x(P_i)$, $\beta_i = y(P_i)$, $i = 1, 2$, and assume without loss of generality that $\beta_1 \geq \alpha_1$. Then the imputation $z$, defined by[7]

$$z = \alpha_1 x_1 + \beta_2 y_2 + (\beta_1 - \alpha_1) u_2,$$

is necessarily in $X$, since any imputation dominating $z$ also dominates either $x$ or $y$. Similarly, the imputation $z'$, defined by

$$z' = \alpha_1 x_1 + (\beta_1 - \alpha_1) u_1 + \beta_2 y_2,$$

is in $X$. But if $\beta_1 - \alpha_1$ were strictly positive, $z$ and $z'$ would dominate each other via the winning coalitions $P_1$ and $P_2$, in violation of the internal stability of $X$. Hence

$$\alpha_1 = \beta_1$$

(making $z$ and $z'$ the same), and we have

$$\alpha_1 x_1 + \alpha_2 y_2 \in X.$$

By symmetry,

$$\alpha_1 y_1 + \alpha_2 x_2 \in X.$$

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[7] We follow our usual convention in regard to ill—defined barycentric projections.

Since $x$ and $y$ were arbitrary points in $X$, these facts show that $X$ is of the required "cartesian" form $X_1 \underset{\alpha}{\times} X_2$, with of course $\alpha = \alpha_1$.

It remains to show that each $X_i$ is, or can be chosen to be, a solution of its corresponding subgame $\Gamma(P_i, \mathcal{W}_i)$. If $\alpha_i = 0$ then $X_i$ is irrelevant to $X$, and we can use any solution we please to satisfy the literal requirements of the theorem. On the other hand, if $\alpha_i > 0$ then $X$ determines a unique $X_i$, and we must prove that the latter is a solution of $\Gamma(P_i, \mathcal{W}_i)$. As before, we do this in two parts. It will suffice to consider the case $i = 1$.

(A) External stability. Let $x_1$, $x_2$ be arbitrary elements of $X_1$, $X_2$, respectively, and take any $y_1 \in A_{P_1} - X_1$. Then the imputation $y$, defined by

$$y = \alpha_1 y_1 + \alpha_2 x_2,$$

is not in $X$, since we are assuming $\alpha_1 > 0$. Hence, some $z \in X$ dominates $y$. Let $S \in \mathcal{W}$ denote the set of players on which $z - y$ is positive. If $S \cap P_2$ were in $\mathcal{W}_2$, then $z$ would dominate $\alpha_1 x_1 + \alpha_2 x_2 \in X$, violating internal stability. Therefore we must have $S \cap P_1 \in \mathcal{W}_1$. Let $z_1 = B_{P_1} z$. Then $z_1 - y_1$ is also positive on $S \cap P_1$. Hence $y_1 \in \mathrm{dom}_1 z_1$. But $z_1$ is clearly an element of $X_1$. Hence $y_1 \in \mathrm{dom}_1 X_1$.

(B) Internal stability. Suppose that $y_1 \in \mathrm{dom}_1 x_1$ for some $x_1$, $y_1 \in X_1$. Then we would also have $\alpha_1 y_1 + \alpha_2 x_2 \in \mathrm{dom}(\alpha_1 x_1 + \alpha_2 x_2)$

for arbitrary $x_2 \in X_2$, since $\mathcal{W}_1 \subset \mathcal{W}$. This contradicts the internal stability of X. This completes the proof of the theorem.

## 4. THE SOLUTIONS OF PRODUCTS: FURTHER RESULTS

The resemblance between sums and products has up to now been so close that only one more step — the product counterpart to Theorem 3 — seems to be needed to "wrap up" the solution theory of these games. Unfortunately, our best efforts in this direction produce not the hoped–for result — which would be a full–fledged converse to Theorem 1 — but only the weaker result given in Theorem 4 below. This states that _if_ a solution of a product is of the form $X_1 \times X_2$, _then_ the sets $X_1$ and $X_2$ are solutions of their respective com–ponents. Though not uninteresting in itself this contributes very little to a complete characterization of the solutions of product games.

The source of the difficulty is not hard to find. Examination of a simple three–person game will reveal a large array of solutions not predicted by Theorem 1. In what proves to be only the first move in exploring the _terra incognita_ of product-game solution theory, we establish a generalization of Theorem 1 (in two slightly different forms), in which continuous variation of the subsolutions $X_1$ and $X_2$ is permitted as a function of the parameter $\alpha$. These results (Theorems 5 and 6 below) dispose of the three–person example that motivated them, but the class of solutions they generate is still far from complete in general, as an example in Sec. 6 will make clear.

A partial converse to Theorem 1.

THEOREM 4. Let $X$ be a solution of $\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$. If sets $X_i \subseteq A_{P_i}$, $i = 1, 2$, exist such that $X = X_1 \times X_2$, then $X_1$ is a solution of $\Gamma(P_1, \mathcal{W}_1)$ and $X_2$ is a solution of $\Gamma(P_2, \mathcal{W}_2)$.

To prove this theorem it will suffice to show that $X_1$ is a solution of $\Gamma(P_1, \mathcal{W}_1)$. (A) External stability. Take $y_1 \in A_{P_1} - X_1$. Then $y_1 \notin X$. Hence $y_1 \in \text{dom } z$ for some $z \in X$. Let $S \in \mathcal{W}$ be the set of players on which $z - y_1$ is positive. Then $S \cap P_1 \in \mathcal{W}_1$. This incidentally gives us $z(P_1) > 0$ so that the barycentric projection of $z$ on $A_{P_1}$ is well defined; denote it by $z_1$. Clearly $z_1 \in X_1$. Moreover, the set on which $z_1 - y_1$ is positive must include $S \cap P_1$, since $z_1 \geq z$ on $P_1$. Therefore we have $y_1 \in \text{dom}_1 z_1 \subseteq \text{dom}_1 X_1$, as required.

(B) Internal stability. Take $y_1 \in X_1$, and suppose that $y_1 \in \text{dom}_1 x_1$ for some $x_1 \in X_1$. Let $S_1 \in \mathcal{W}_1$ denote the set of players on which $x_1 - y_1$ is positive. We assert (proof below) that we can find $x_2 \in X_2$ and $S_2 \in \mathcal{W}_2$ such that $x_2$ is strictly positive on $S_2$. The imputation $x$, defined by

$$x = (1 - \varepsilon)x_1 + \varepsilon x_2,$$

is then in $X$, and for small $\varepsilon > 0$ the difference $x - y_1$ is positive on both $S_1$ and $S_2$. Hence $y_1 \in \text{dom } X$. Since $y$ is an element of $X$, this violates internal stability of $X$, and refutes the original assumption that $y_1 \in \text{dom}_1 X_1$.

° The assertion in the third sentence of (B) must still be proved.

Part (A) permits us to assume that $X_2$ is at least externally stable. In particular, $A_{P_2} \cap dom_2 X_2$ is not empty unless $X_2 = A_{P_2}$. If the latter, we may take $S_2 = P_2$ and $x_2$ any interior point of $A_{P_2}$. Otherwise, let $z_2$ by any point in $A_{P_2} \cap dom_2 X_2$, let $x_2$ be a point of $X_2$ that dominates $z_2$, and let $S_2$ be the set on which $x_2 - z_2$ is positive. In both instances we obtain $x_2 \in X_2$ and $S_2 \in \mathcal{W}_2$ such that $x_2$ is strictly positive on $S_2$, as required. This completes the proof of the theorem.

**A simple example.** In order to motivate the next definition, let us consider a particular three-person compound game "G":[8]

$$G = \Gamma(\overline{123}, \ \{\overline{13}, \ \overline{23}, \ \overline{123}\}).$$

This decomposes into a product, one term of which is a sum:

$$G = (B_1^1 \oplus B_1^2) \otimes B_1^3.$$

Here $B_1^i$ denotes the one-person simple game $\Gamma(\{i\}, \ \{\{i\}\})$. The solutions of $G$ are well known. They include all the straight lines in the triangle of imputations joining the vertex $A_{\overline{3}}$ to the opposite edge $A_{\overline{12}}$, i.e., the solutions predicted by Theorem 1. They also include all other arcs running monotonically from $A_{\overline{3}}$ to $A_{\overline{12}}$, curves along which the share

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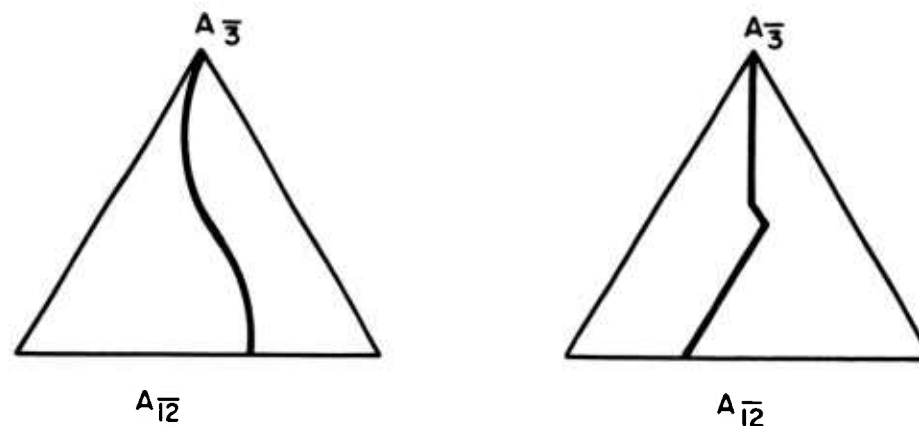[8] For brevity, we write $\overline{123}$ for $\{1, 2, 3\}$, etc.

Fig. 2 —— Typical solutions of $(B_1^1 \oplus B_1^2) \otimes B_1^3$

of player 3 decreases steadily from 1 to 0 while the shares of players 1 and 2 increase or remain constant (see Fig. 2). There are no other solutions.

The existence of these curvilinear solutions shows that our method of combining subsolutions by means of the "join" operation has been needlessly rigid. It would seem that the relaxation indicated is to permit the sub—solutions to vary as a function of the parameter $\alpha$ that appears in the definition of "$\times$", but they must not be permitted to vary too rapidly. How best to control the variation when the subsolutions are sets rather than points (as above) is a rather delicate problem. The method we adopt first is not the only possible one, but it does lead to a substantial

generalization of Theorem 1 and, in the process, enables us to account for all the solutions of the present example.

Semimonotonic families.   A parametrized family of sets of imputations:

$$\{X(\alpha) \mid 0 \leq \alpha \leq 1\}$$

will be called semimonotonic if for every $\alpha$, $\beta$, x such that $0 \leq \beta \leq \alpha \leq 1$ and $x \in X(\alpha)$, there exists a $y \in X(\beta)$ such that $\beta y \leq \alpha x$.

One example of a semimonotonic family is a constant family: $X(\alpha) \equiv X$. Another is a family that is monotonically decreasing in the sense of set—inclusion: $\alpha > \beta$ implies $X(\alpha) \subseteq X(\beta)$.   A third example can be derived from an arbitrary solution of the game G  just discussed, by taking $X(\alpha)$ to be the barycentric projection of the single point comprising the "$\alpha$" cross—section of the solution.[9]

A generalization of Theorem 1.

THEOREM 5.  Let $\{X_i(\alpha_i) \mid 0 \leq \alpha_i \leq 1\}$  be a semimonotonic family of solutions of the game $\Gamma(P_i, \mathcal{W}_i)$, for $i = 1, 2$.  Then the set

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha)$$

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[9]Our terminology is less than ideal, since the vectors in each $X(\alpha)$ must be multiplied by $\alpha$ before anything monotonic in the ordinary sense emerges.  The "semi" refers to the one—sideness of the hypothesis $\beta \leq \alpha$ (compare the definition of "monotonic" later in this section).

is a solution of the product game

$$\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2).$$

The proof that follows is parallel in most respects to that of Theorem 1. (A) External stability. Take any $y \in A_P$ and define $\beta_i = y(P_i)$, $i = 1, 2$.

Case 1: $0 < \beta_1 < 1$.

Case 1a: $y_1 \in X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$. Then $y \in X$.

Case 1b: $y_1 \notin X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$. Find $z_1 \in X_1(\beta_1)$ and $z_2 \in X_2(\beta_2)$ such that $y_1 \in \text{dom}_1 z_1$ and $y_2 \in \text{dom}_2 z_2$. Then the imputation $z$ defined by

$$z = \beta_1 z_1 + \beta_2 z_2,$$

which is in $X$, clearly dominates $y$. Hence $y \in \text{dom } X$.

Case 1c: $y_1 \in X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$. Find $z_2 \in X_2(\beta_2)$ such that $y_2 \in \text{dom}_2 z_2$. Let $S_2 \in \mathcal{W}_2$ be the set of players on which $z_2 - y_2$ is positive. Choose $\mathcal{E} > 0$ so that $\beta_2 z_2 - \beta_2 y_2$ is greater than $\mathcal{E}$ on $S_2$. Using the semimonotonic property of $\{X_2(\alpha)\}$, find $z_2'$ in $X_2(\beta_2 - \mathcal{E})$ such that $\beta_2 z_2 - (\beta_2 - \mathcal{E})z_2'$ is nonnegative. This vector must be $\leq \epsilon$ in all components, since no component of a nonnegative vector can exceed the sum of all components. It follows that we have

$$(\beta_2 - \mathcal{E})z_2' > \beta_2 y_2 \text{ on } S_2.$$

Let $u_1$ be any interior point of $A_{P_1}$. Then the imputation $x$ defined by

$$x = \beta_1 y_1 + \varepsilon u_1 + (\beta_2 - \varepsilon)z_2'$$

dominates $y$, because $x - y$ is positive on the winning set $P_1 \cup S_2$. If $x \in X$ then $y \in \operatorname{dom} X$, and we are finished with this case. Suppose $x \notin X$. Then it must be the barycentric projection of $x$ on $A_{P_1}$, namely

$$x_1 = \frac{\beta_1}{\beta_1 + \varepsilon} y_1 + \frac{\varepsilon}{\beta_1 + \varepsilon} u_1,$$

that fails to lie in the corresponding subsolution $X_1(\beta_1 + \varepsilon)$. Hence we can find $z_1 \in X_1(\beta_1 + \varepsilon)$ such that $x_1 \in \operatorname{dom}_1 z_1$. Then it is clear that the imputation $z$, defined by

$$z = (\beta_1 + \varepsilon)z_1 + (\beta_2 - \varepsilon)z_2',$$

dominates $y$. Moreover $z \in X$. Hence $y \in \operatorname{dom} X$.

Case 1d: $y_1 \notin X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$. Like Case 1c.

Case 2: $\beta_1 = 0$.

Case 2a: $y_2 \in X_2(1)$. Then $y \in X$.

Case 2b: $y_2 \notin X_2(1)$. Then the argument of Case 1c may be repeated, with the usual understanding that $\beta_1 y_1$ is zero.

Case 3: $\beta_1 = 1$. Like Case 2. This completes part (A) of the proof.

(B) Internal stability.  Suppose that for some  $x, y \in X$,  $x$  dominates  $y$  — i.e.,  $x - y$  is positive on a winning set  $S \in \mathcal{W}$.  With no loss of generality we may assume that  $y(P_1) \geq x(P_1)$.  Define  $\alpha_1 = x(P_1)$,  $\beta_1 = y(P_1)$,  $S_1 = S \cap P_1 \in \mathcal{W}_1$.  We have

$$\beta_1 \geq \alpha_1 \geq x(S_1) > y(S_1) \geq 0;$$

hence, the projections  $x_1 = B_{P_1} x$,  $y_1 = B_{P_1} y$  are well defined.  More-over,  $x_1 \in X_1(\alpha_1)$  and  $y_1 \in X_1(\beta_1)$.  Using the semimonotonicity of  $\{X_1(\alpha)\}$,  we can find  $z_1 \in X_1(\alpha_1)$  such that  $\alpha_1 z_1 \leq \beta_1 y_1$.  But by our original assumption,  $\alpha_1 x_1 - \beta_1 y_1$  is positive on  $S_1$.  Hence  $z_1 \in \mathrm{dom}_1 x_1$, contrary to the internal stability of  $X_1(\alpha_1)$.  This contradiction establishes the internal stability of  $X$,  and completes the proof of the theorem.

Monotonic families and a variant of Theorem 5.  The alert reader may have observed that the internal stability of the sets  $X_i(1)$,  $i = 1$,  2, was not used in the preceding proof.  (Note that the  $\alpha_1$  in (B) is always less than 1, since  $\alpha_2$  is necessarily positive.)  This observation by itself does not enlarge the scope of the theorem, since the internal stability of nearby sets  $X_i(1 - \mathcal{E})$  implies the internal stability of  $X_i(1)$  as well, whether we want it or not.  To obtain a real extension of the theorem in this direction, we must tighten some of the other hypotheses.  Because of this adjustment, Theorem 6 will be independent of Theorem 5:  each yields some solutions that the other misses.

A semimonotonic family $\{Y(\alpha) \mid 0 \le \alpha \le 1\}$ will be called

monotonic if for every $\alpha$, $\beta$, x such that $0 \le \alpha \le \beta \le 1$ and $x \in Y(\alpha)$,

there exists a $y \in Y(\beta)$ such that $\beta y \ge \alpha x$. (Compare the previous

definition of "semimonotonic," p. 22.)

THEOREM 6. Let $\{Y_i(\alpha_i)\}$ be a monotonic family of solutions of

$\Gamma(P_i, \mathcal{N}_i)$, except that $Y_i(1)$ need not be externally stable, and define

$$X_i(\alpha_i) = A_{P_i} - \text{dom}_i Y_i(\alpha_i),$$

all for $i = 1, 2$. Then the set:

$$X = \bigcup_{0 \le \alpha \le 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1 - \alpha)$$

is a solution of $\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$.

Note that the sets $X_i$ and $Y_i$ are identical except when $\alpha_i = 1$. In

the latter case, $X_i$ is obtained from $Y_i$ by adding the external points

that $Y_i$ fails to dominate, if any. Thus, $X_i(1)$ must be externally stable,

but is perhaps not internally stable; moreover, it may contain points that

are not monotonically related to the nearby sets $X_i(1 - \mathcal{E})$. (For an

illustration, see Fig. 4c on p. 32.)

Theorem 6 is certainly not a major extension of the theory; it does

give us, however, a first indication of the occurrence of nonsolutions in

the cross sections of a solution of a product game. The proof, which we

omit, entails only a slight modification of the one preceding, and should give no trouble to the aforementioned alert reader.

In the next section we shall produce specific examples that demonstrate the logical independence of Theorems 5 and 6. In one case we shall find a semimonotonic, nonmonotonic family of solutions to a certain game J. This means that games of the form $J \circledtimes K$ will have solutions that are given by Theorem 5 but not Theorem 6. In another case we shall find an application of Theorem 6 in which $X_i(1)$ and $Y_i(1)$ are different, and which therefore yields solutions not given by Theorem 5.

In conclusion, we point out that the word "monotonic" in Theorem 6 cannot be replaced by "semimonotonic," since this would permit the deletion of an arbitrary subset of $Y_i(1)$, and lead quickly to a counter-example. Whether the requirement of full monotonicity can be relaxed outside a neighborhood of $\alpha_i = 1$ is not known at present. If it can, then a nontrivial generalization including both theorems might result.

## 5. SOLUTIONS OF SOME SPECIFIC PRODUCT GAMES

The pursuit of the complexities arising in the solutions of product games is justified in part by the insights this activity provides into the still more difficult problems surrounding the solutions of general n–person games. Our results thus far in this paper (on product games) are primarily of a constructive nature. The examples to be given in this section illustrate some conspicuous limitations of our present techniques. They are intended partly to guide further work on product games and partly to

stimulate efforts to find effective ways of using other general properties
of the game structure to build up solutions.

Our first example is a four—person game "H":

$$H = \Gamma(P, \mathcal{H}) = \Gamma(\overline{1234}, \ \{\overline{134}, \ \overline{234}, \ \overline{1234}\}).$$

This game is a product of three components, namely $B_1^1 \oplus B_1^2$, $B_1^3$,
and $B_1^4$. Denote the first by $B_2^*$. The factors can be associated in two
essentially different ways, as follows:

.

(a) $H = B_2^* \otimes \left( B_1^3 \otimes B_1^4 \right)$,

(b) $H = \left( B_2^* \otimes B_1^3 \right) \otimes B_1^4$.

We shall discover that our methods give different sets of solutions to H,
depending on which grouping is used.

In the first case (a) there is no difficulty in solving completely the
two two—person factors. A solution of $B_2^*$ is an arbitrary single point of
$A_{\overline{12}}$, while the unique solution of $B_1^3 \otimes B_1^4$ is the whole set $A_{\overline{34}}$.
Application of Theorem 5 (or 6) yields the class of product solutions
exemplified in Fig. 3a. They are three—cornered surfaces spanning the
tetrahedron $A_{\overline{1234}}$, made up of linear elements parallel to the edge
$A_{\overline{34}}$. (The dashed lines in the figure outline a typical cross—section of
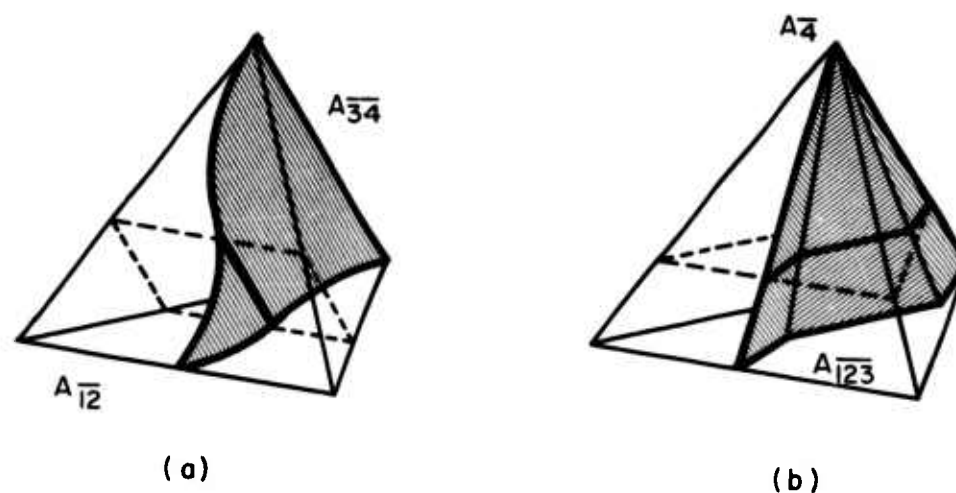
Fig. 3 — Typical solutions of $(B_1^1 \oplus B_1^2) \otimes B_1^3 \otimes B_1^4$

constant $\alpha$.) The curvature of the surface is thus restricted to one dimension, and is further constrained by the monotonicity condition . Verbally, the latter requires that the payoffs to players 1 and 2 must both increase (or remain constant) if the combined payoff to players 3 and 4 goes down.

Now consider the grouping (b). It consists of the one-person game $B_1^4$ multiplied by a three-person game which we have already considered: the game "G" of Sec. 4 (see Fig. 2). Theorem 5 (or 6) again yields a class of three-cornered surfaces spanning the simplex $A_{\overline{1234}}$, but they are generated this time by arbitrary, monotonically-varying, monotonic curves in the cross sections parallel to the base $A_{\overline{123}}$. The curvature of the surface will in general be two-dimensional,

but in Fig. 3b we have been content to depict the conical type of surface

(vertex at $A_{\overline{4}}$) that results from an application of Theorem 1.

A comparison of the two classes of solutions reveals that "(b)"

includes "(a)" but not vice versa. Class "(b)" in fact contains all the

solutions of the game (we omit the proof). It is the failure of "(a)" to

do the same that is the provocative feature of this example. The trouble

seems to be that most of the solutions (including the one shown in Fig. 3b)

simply do not decompose in accordance with the partition $\{\overline{12}, \overline{34}\}$.

Instead, the distributions of proceeds inside the two subgames are inter—

dependent. There seems no reason to believe that this phenomenon will

not be found in other product games, where there may be no alternative

arrangement of factors to save the day.

A final example. The four—person game that we shall call "J" is

defined by

$$J = \Gamma(\overline{1234}, \ \{\overline{124}, \ \overline{134}, \ \overline{234}, \ \overline{1234}\}).$$

It differs from the preceding "H" only in the possession of one additional

winning coalition. Since the game is weak (see p. 5), the "veto" player

can be factored out, and we obtain the decomposition

$$J = M_3 \otimes B_1^4.$$

Here $M_3$ denotes the simple majority game on the set $\overline{123}$. The complete list of solutions to $M_3$ is well known;[10] in our present normalization it may be rendered as follows:

(a)    The finite set $\left\{ \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right) \right\}$.

($b_\delta$)    The line-segment $\{(\delta, t, 1-\delta-t) \mid 0 \le t \le 1-\delta\}$, for each $0 \le \delta < \frac{1}{2}$.

($c_\delta$)    The line-segment obtained by permuting players 1 and 2 in ($b_\delta$), for each $0 \le \delta < \frac{1}{2}$.

($d_\delta$)    The line-segment obtained by permuting players 1 and 3 in ($b_\delta$), for each $0 \le \delta < \frac{1}{2}$.

No semimonotonic family drawn from this list can include representatives from more than one of the four categories (a) – (d). The only possible variation within such a family is in the value of $\delta$, and semimonotonicity in fact implies monotonicity. Theorem 5 therefore generates only solutions of the types exemplified in Figs. 4a and 4b.

Theorem 6, for once, gives us something new. Setting $\delta = 1/2$ in (b), (c), or (d) above produces an internally stable set (not a solution of $M_3$) that is monotonically related to the solutions nearby, and is therefore
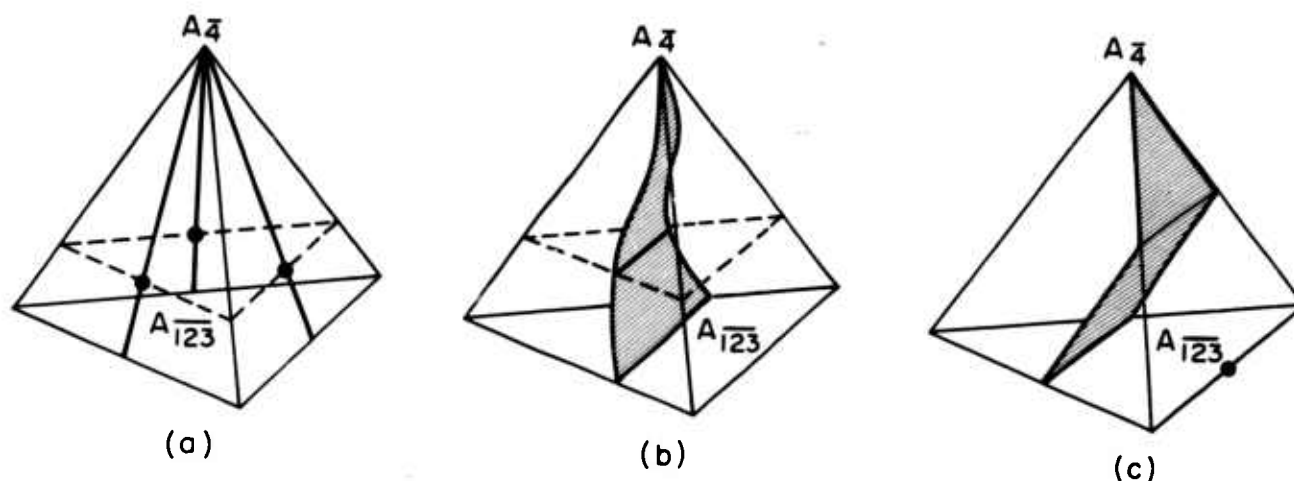
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[10]See [5], page 282 ff.

Fig. 4 —Typical solutions of $M_3 \otimes B_I$

available as a "$Y_i(1)$" in the application of Theorem 6. Figure 4c illus—
trates one of the resulting solutions of J; its intersection with the base
of the tetrahedron, we see, is internally <u>unstable</u> in the factor $M_3$.
(The isolated point has coordinates (0, 1/2, 1/2, 0).) This is one of the
examples promised at the end of Sec. 4.

Continuing, we observe that the solution of Fig. 4c, without the
isolated point, can be represented as a limit of solutions like that of
Fig. 4b. This fact enables us to construct a parametrized family of
solutions of the four—person game J that is semimonotonic, but not
monotonic. For example, we can make $X(\alpha)$ like Fig. 4c for $\alpha \leq 1/2$

and like Fig. 4b for $\alpha > 1/2$, keeping everything perfectly continuous and monotonic except for the abrupt disappearance of the isolated point at $\alpha = 1/2$. This family can then be used to produce solutions to games of the form $J \otimes K$ — solutions which Theorem 5 would predict, but not Theorem 6. This is the other example promised at the end of Sec. 4.

The game "J," innocent as it appears, actually has an enormous variety of solutions. It has been shown [2] that one may choose an entirely arbitrary closed subset of the line connecting $A_{\overline{4}}$ to the mid-point of $A_{\overline{23}}$ (or $A_{\overline{12}}$ or $A_{\overline{13}}$), and construct a solution containing this set and no other nearby points, except in the neighborhood of $A_{\overline{4}}$. Figure 5 illustrates the construction. [11]

Still other solutions exist to this game. For example, it is possible to twist a solution so as to rotate the "odd man's" role among players $1-3$ at different levels. In particular, it is possible to make all of the categories (a) — (d) listed on page 31 appear as cross sections of the same solution. A limit to the complexity of this game may be in view, however, since the possible cross sections of its solutions appear to be restricted to the six types shown in Fig. 6, and their rotations. Three of these types represent nonsolutions of $M_3$: specifically "c" is internally unstable and "e" and "f" are externally unstable. All six types occur in the solution illustrated in Fig. 5: reading top to bottom the sequence is a, b, c, d, e, b, c, e, f, e.

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[11] The figure is reproduced from [2]. The arbitrary set here consists of the "dash" and two "dots" lying on the indicated median of $A_{\overline{234}}$.
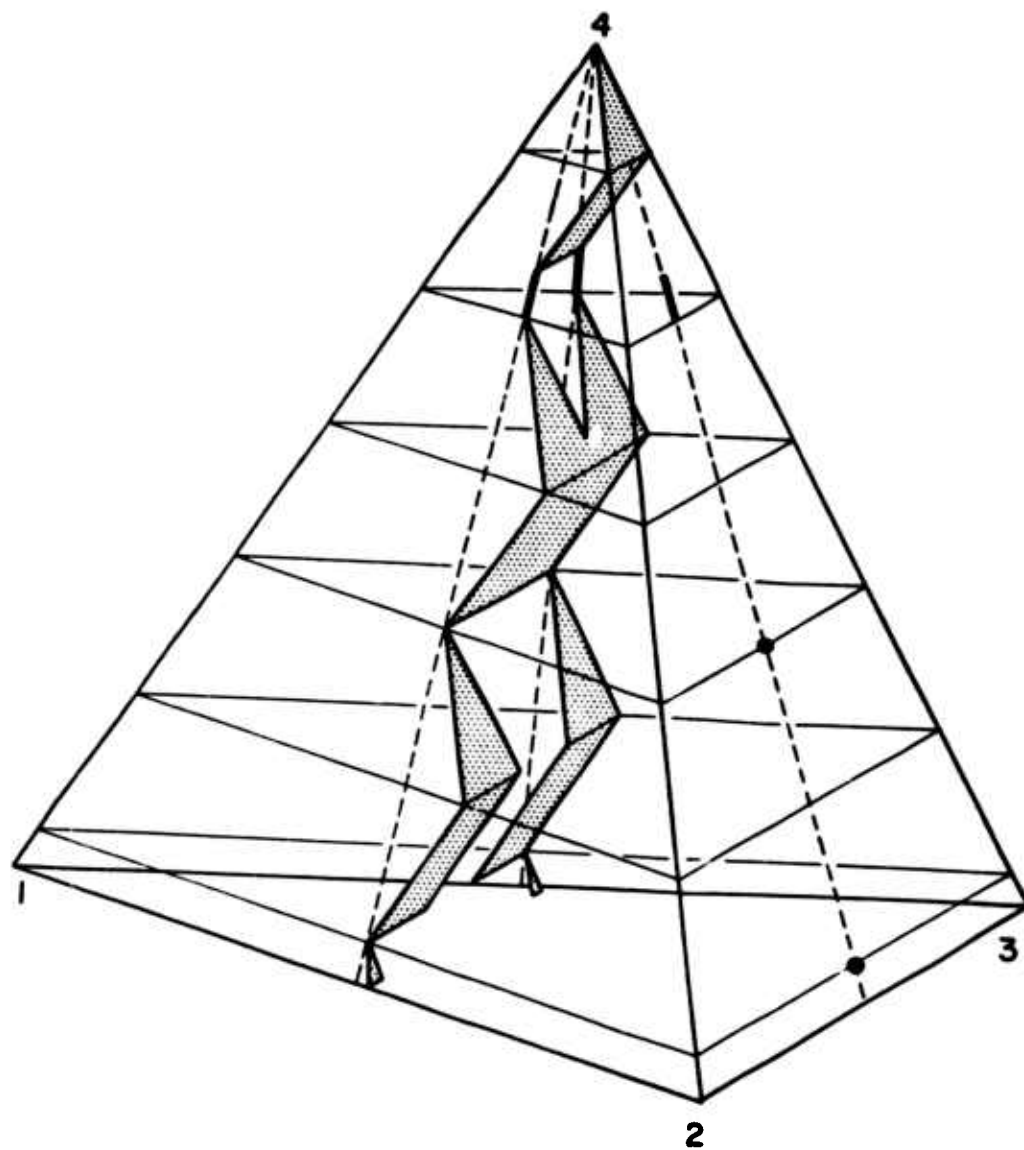
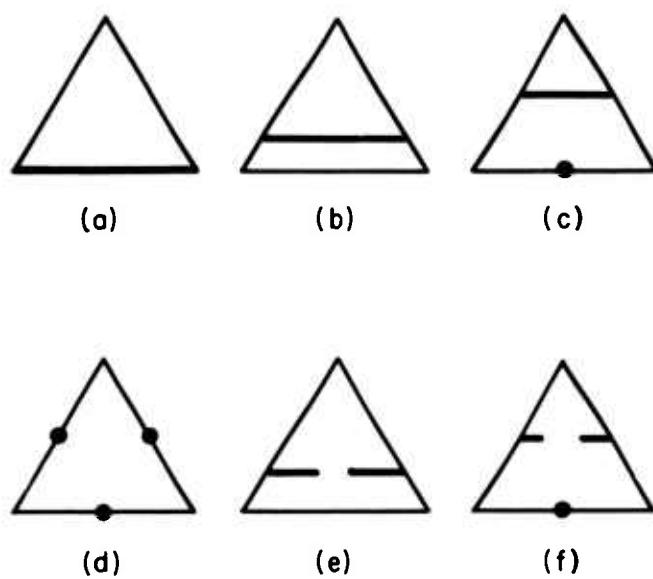Fig. 5 — An irregular solution of $M_3 \otimes B_1$

Fig. 6— Cross-section types for $M_3 \otimes B_1$

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